

# An Abstract Extremal Principle with Applications to Welfare Economics

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In this paper we introduce general prenormal and normal structures in Banach spaces that cover conventional concepts of normals to arbitrary closed sets under minimal requirements. Based on these structures, we establish new abstract versions of the extremal principle in variational analysis, which plays a fundamental role in many applications. The main applications of this paper concern necessary conditions for Pareto optimality in nonconvex models of welfare economics. We obtain new results in this direction that extend approximate and exact versions of the generalized second welfare theorem for Pareto, weak Pareto, and strong Pareto optimal allocations. © 2000 Academic Press

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## 1. INTRODUCTION

It has been well recognized that the convex separation principle plays a crucial role in many aspects of nonlinear analysis, optimization, and their applications. In particular, a conventional approach to derive necessary optimality conditions in various optimization, optimal control, and equilibrium problems consists of applying convex separation theorems to either the convex sets in question or their tangential convex approximations.

This paper develops another approach to optimal solutions and related aspects of variational analysis that does not involve any convex approxima-

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tions and convex separation arguments. Instead, it is based on a different principle to study extremality of set systems using (generally nonconvex) normal cones in dual spaces that are not generated by primal tangential approximations. This approach, unified under the name of the *extremal principle* [27], goes back to the beginning of dual-spaced methods in nonsmooth variational analysis; see [26, 28] for more details, references, and discussions. Results obtained in this direction can be treated as variational extensions of the classical separation theorems to systems of nonconvex sets.

The primary goal of this paper is to obtain general versions of the extremal principle in terms of abstract prenormal and normal structures in Banach spaces. Then we apply these results to the study of Pareto optimal allocations in nonconvex models of welfare economics. Discussions of the results obtained and their comparison with the literature are presented in the subsequent sessions.

Our notation is basically standard. Let us mention that  $B \subset X$  and  $B^* \subset X^*$  stand, respectively, for the unit closed balls in the Banach space in question and its dual;  $\xrightarrow{w^*}$  signifies the weak\* convergence in  $X^*$ , and  $\text{cl}^*$  denotes the weak\* topological closure. Depending on the context, we use the notation  $\text{Lim sup}$  for either the *topological* Painlevé–Kuratowski upper (outer) limit

$$\text{Lim sup}_{x \rightarrow \bar{x}} F(x) := \text{cl}^* \left\{ x^* \in X^* \mid \exists \text{ sequences } x_k \rightarrow \bar{x}, x_k^* \xrightarrow{w^*} x^* \right. \\ \left. \text{with } x_k^* \in F(x_k), k \in \mathbb{N} \right\} \quad (1.1)$$

of a set-valued mapping  $F: X \rightrightarrows X^*$ , or for its *sequential* counterpart when  $\text{cl}^*$  is omitted in (1.1).

## 2. NORMAL STRUCTURES IN BANACH SPACES

In this section we consider abstract concepts of normals to arbitrary subsets of Banach spaces and designate minimal requirements to such concepts that allow us to derive fuzzy and exact versions of a general extremal principle, which is the main tool of our analysis and applications to welfare economics.

**DEFINITION 2.1.** Let  $X$  be a Banach space. We say that  $\widehat{N}$  defines a *prenormal structure* in  $X$  if it associates, with every nonempty closed set

$\Omega \subset X$ , a set-valued mapping  $\widehat{N}(\cdot; \Omega): X \rightrightarrows X^*$  such that  $\widehat{N}(x; \Omega) = \emptyset$  for  $x \notin \Omega$  and the following property holds:

(H1) Given any small  $\varepsilon > 0$ ,  $a \in X$  with  $\|a\| \leq \varepsilon$ , and closed sets  $\Omega_1, \Omega_2 \subset X$ , assume that  $(\bar{x}_1, \bar{x}_2) \in \Omega_1 \times \Omega_2$  is a local minimizer of the function

$$g(x_1, x_2) := \|x_1 - x_2 + a\| + \varepsilon(\|x_1 - \bar{x}_1\| + \|x_2 - \bar{x}_2\|) \quad (2.1)$$

relative to the set  $\Omega_1 \times \Omega_2$  with  $\bar{x}_1 - \bar{x}_2 + a \neq 0$ . Then there are  $\tilde{x}_i \in \bar{x}_i + \varepsilon B$ ,  $i = 1, 2$ , and  $x^* \in X^*$  with  $\|x^*\| = 1$  such that

$$(-x^*, x^*) \in \widehat{N}(\tilde{x}_1; \Omega_1) \times \widehat{N}(\tilde{x}_2; \Omega_2) + \gamma(B^* \times B^*) \quad \text{for all } \gamma > \varepsilon. \quad (2.2)$$

Property (H1) postulates an ability of the prenormal structure  $\widehat{N}$  to describe *first-order necessary optimality conditions* for minimizing functions of the norm type (2.1) over arbitrary closed sets. Note that (2.2) provides a “fuzzy” optimality condition since it involves points  $(\tilde{x}_1, \tilde{x}_2)$  close to the given minimizer with  $\gamma > \varepsilon$  in (2.2).

Let us show that property (H1) always holds for *subdifferentially generated* prenormal structures with a minimal set of natural requirements. We say that  $\hat{\delta}$  is a *presubdifferential* on the class of *lower semicontinuous* extended-real-valued functions  $f: X^2 \rightarrow (-\infty, \infty]$  with  $\text{dom } f := \{x | f(x) < \infty\}$  if  $\hat{\delta}f(x) = \emptyset$  for  $x \notin \text{dom } f$  and the following properties hold:

(S1) Suppose that  $\bar{x}$  is a local minimizer of the sum of two functions  $g + h$  finite at  $\bar{x}$ , where  $g$  is a convex continuous function of type (2.1) and  $h$  is a lower semicontinuous function of the set indicator type. Then for any  $\varepsilon > 0$  there are  $u, v \in \bar{x} + \varepsilon B$  such that  $h(v) \leq h(\bar{x}) + \varepsilon$  and

$$0 \in \hat{\delta}g(u) + \hat{\delta}h(v) + \varepsilon B^*.$$

(S2) If  $g$  is convex continuous of type (2.1), then  $\hat{\delta}g(\cdot)$  agrees with the subdifferential of  $g$  in the sense of convex analysis.

(S3) If  $f(x_1, x_2) = f_1(x_1) + f_2(x_2)$ , then  $\hat{\delta}f(\bar{x}_1, \bar{x}_2) \subset \hat{\delta}f_1(\bar{x}_1) \times \hat{\delta}f_2(\bar{x}_2)$  for any  $\bar{x}_i \in \text{dom } f_i$ ,  $i = 1, 2$ .

Note that we use the word “presubdifferential” instead of “subdifferential” for constructions  $\hat{\delta}$  satisfying (S1)–(S3) since we expect more elaborated properties from subdifferentials, in particular, the “exact” calculus rule ( $\varepsilon = 0$ ) in (S1) instead of the “fuzzy” one with  $\varepsilon > 0$ . Similarly, the construction  $\widehat{N}(x; \Omega) := \hat{\delta}\delta(\bar{x}; \Omega)$  generated by a presubdifferential of the set indicator function  $\delta(\cdot; \Omega)$  is called a *prenormal cone* of  $\Omega$  at  $\bar{x} \in \Omega$ .

It is well recognized that the requirements (S1)–(S3), together with other natural properties, are satisfied for most known subdifferential constructions in appropriate Banach spaces; see, e.g., [17] and the references therein. Let us show that these minimal subdifferential requirements are sufficient to imply the basic property (H1) of the corresponding prenormal cones. In what follows, we always use the sum norm

$$\|(x_1, x_2)\| := \|x_1\| + \|x_2\| \quad (2.3)$$

on  $X^2$ . In this case, the dual norm on  $(X^2)^*$  is given by

$$\|(x_1^*, x_2^*)\| = \max\{\|x_1^*\|, \|x_2^*\|\}.$$

**PROPOSITION 2.2.** *Let  $\widehat{N}(\cdot; \Omega) := \hat{\partial}\delta(\cdot; \Omega)$  be a cone generated by a presubdifferential  $\hat{\partial}$  satisfying (S1)–(S3). Then  $\widehat{N}(\cdot; \Omega)$  defines a prenormal structure in  $X$ ; i.e., it has property (H1).*

*Proof.* Consider the function  $g(x_1, x_2)$  in (H1) with given  $\bar{x} := (\bar{x}_1, \bar{x}_2)$ ,  $\varepsilon > 0$ , and  $a \in X$  satisfying  $\nu := \|\bar{x}_1 - \bar{x}_2 + a\| > 0$ . Observe that  $\bar{x}$  is a local minimizer of the function

$$f(x_1, x_2) := g(x_1, x_2) + \delta((x_1, x_2); \Omega_1 \times \Omega_2) \quad (2.4)$$

over  $X^2$  with no additional constraints. Take any  $\gamma > \varepsilon$  and put  $\tilde{\varepsilon} := \gamma - \varepsilon$ . We can always select  $\tilde{\varepsilon}$  so that

$$0 < \tilde{\varepsilon} \leq \min\{\varepsilon, \nu/2\}. \quad (2.5)$$

Applying (S1) with  $\varepsilon := \tilde{\varepsilon}$  to the sum in (2.4) and using the definition of  $\widehat{N}$  admitted in this proposition, we find  $u = (\hat{x}_1, \hat{x}_2) \in X^2$  and  $v = (\tilde{x}_1, \tilde{x}_2) \in \Omega_1 \times \Omega_2$  such that

$$\max\{\|\hat{x}_1 - \bar{x}\|, \|\hat{x}_2 - \bar{x}_2\|, \|\tilde{x}_1 - \bar{x}_1\|, \|\tilde{x}_2 - \bar{x}_2\|\} \leq \tilde{\varepsilon} \leq \varepsilon \quad (2.6)$$

and

$$\begin{aligned} 0 \in \hat{\partial}\{\|\cdot - \cdot + a\| + \varepsilon(\|\cdot - \bar{x}_1\| + \|\cdot - \bar{x}_2\|)\}(\hat{x}_1, \hat{x}_2) \\ + \widehat{N}((\tilde{x}_1, \tilde{x}_2); \Omega_1 \times \Omega_2) + \tilde{\varepsilon}B_{X^2}^*. \end{aligned} \quad (2.7)$$

By the choice of  $\tilde{\varepsilon} \leq \nu/2$  and estimate (2.6), we get

$$\|\hat{x}_1 - \hat{x}_2 + a\| \geq \|\bar{x}_1 - \bar{x}_2 + a\| - (\|\hat{x}_1 - \bar{x}\| + \|\hat{x}_2 - \bar{x}_2\|) = \nu - 2\tilde{\varepsilon} > 0. \quad (2.8)$$

Observe also that (S3) yields

$$\widehat{N}((\tilde{x}_1, \tilde{x}_2); \Omega_1 \times \Omega_2) \subset \widehat{N}(\tilde{x}_1; \Omega_1) \times \widehat{N}(\tilde{x}_2; \Omega_2).$$

By (S2) the presubdifferential  $\hat{g}(\cdot)$  of the convex function (2.1) is the subdifferential of convex analysis. So we can freely use in (2.7) well-known subdifferential results of convex analysis including the sum rule and subdifferential formulas for normal functions and compositions of the type  $\|Ay + a\|$ , where  $A: X^2 \rightarrow X$  is a linear bounded operator onto  $X$ . This gives

$$\hat{g}(\|\cdot - \cdot + a\|)(\hat{x}_1, \hat{x}_2) = (x^*, -x^*) \quad \text{with } \|x^*\| = 1$$

due to (2.8) and

$$\hat{g}\{\varepsilon(\|\cdot - \bar{x}_1\| + \|\cdot - \bar{x}_2\|)\}(\hat{x}_1, \hat{x}_2) \subset \varepsilon(B_X^* \times B_X^*) = \varepsilon B_{X^2}^*$$

due to (2.3). Combining all these calculations and taking into account that  $\tilde{\varepsilon} = \gamma - \varepsilon$ , we deduce inclusion (2.2) from the one in (2.7) and complete the proof of the proposition. ■

The result obtained provides an important example of a prenormal structure given by subdifferentially generated conic sets. Observe that generally the  $\widehat{N}(x; \Omega)$  in (H1) do *not need* to be cones or any unbounded sets in order to describe conditions of form (2.2). Indeed, the constraint  $\|x^*\| = 1$  does not require any unboundedness of the right-hand side set in (2.2). Note also that a prenormal structure  $\widehat{N}$  does not need to be subdifferentially generated.

Let us describe another class of prenormal structures in  $X$  associated with presubdifferentials of *distance functions* under minimal requirements. Take any number  $L > 1$  and consider an arbitrary presubdifferential  $\hat{g}$  on the class of *Lipschitz continuous* functions  $f: X^2 \rightarrow \mathbb{R}$  satisfying (S2) and the following requirements:

(S1') Property (S1) holds, where  $h$  is a Lipschitz continuous function with modulus  $L$ .

(S3') Property (S3) holds, where  $f_1$  and  $f_2$  are Lipschitz continuous functions with modulus  $L$ .

(S4)  $\hat{g}f(x) = \hat{g}g(x)$  if  $f$  and  $g$  coincide in some neighborhood of  $x$ .

Given  $L > 1$  and a presubdifferential  $\hat{g}$ , for every closed set  $\Omega \subset X$  we define a set-valued mapping  $\widehat{N}(\cdot; \Omega): X \rightrightarrows X^*$  as

$$\widehat{N}(x; \Omega) := \begin{cases} \hat{g}(L \text{ dist}(x; \Omega)) & \text{if } x \in \Omega, \\ \emptyset & \text{otherwise.} \end{cases} \quad (2.9)$$

Note that all the sets  $\widehat{N}(x; \Omega)$  are bounded for reasonable presubdifferential constructions, although we do not need this property in what follows. The next proposition shows that (2.9) defines a prenormal structure in  $X$  under the only requirements (S1'), (S2), (S3'), and (S4).

**PROPOSITION 2.3.** *Let  $\hat{\partial}$  be an arbitrary presubdifferential satisfying the requirements (S1'), (S2), (S3'), and (S4) with some  $L > 1$ . Then  $\widehat{N}$  given in (2.9) defines a prenormal structure in  $X$ ; i.e., it has property (H1).*

*Proof.* Let us show that property (H1) holds for (2.9) if  $\varepsilon > 0$  is sufficiently small. Fix  $L > 1$  and take  $0 < \varepsilon \leq (L - 1)/2$ . Since  $(\bar{x}_1, \bar{x}_2) \in \Omega_1 \times \Omega_2$  is a local minimizer of the function  $g(x_1, x_2)$  in (2.1) over the set  $\Omega_1 \times \Omega_2$ , we find neighborhoods  $U_1$  of  $\bar{x}_1$  and  $U_2$  of  $\bar{x}_2$  such that  $g$  attains its global minimum over  $(\Omega_1 \cap U_1) \times (\Omega_2 \cap U_2)$  at  $(\bar{x}_1, \bar{x}_2)$ . It is easy to see that  $g$  is Lipschitz continuous on  $X^2$  with modulus  $1 + 2\varepsilon \leq L$ . Using [9, Proposition 2.4.3], we conclude that the function

$$f(x_1, x_2) := g(x_1, x_2) + L \operatorname{dist}((x_1, x_2); (\Omega_1 \cap U_1) \times (\Omega_2 \cap U_2)) \quad (2.10)$$

attains its minimum over the whole space  $X^2$  at  $(\bar{x}_1, \bar{x}_2)$ . It follows from (2.3) that

$$\begin{aligned} & \operatorname{dist}((x_1, x_2); (\Omega_1 \cap U_1) \times (\Omega_2 \cap U_2)) \\ &= \operatorname{dist}(x_1; \Omega_1 \cap U_1) + \operatorname{dist}(x_2; \Omega_2 \cap U_2). \end{aligned} \quad (2.11)$$

Similarly to the proof of Proposition 2.2, we pick  $\gamma > 0$ ,  $\nu := \|\bar{x}_1 - \bar{x}_2 + a\|$  and select  $\tilde{\varepsilon} := \gamma - \varepsilon$  satisfying (2.5). Employing property (S1') for the sum of functions in (2.10) with  $\varepsilon := \tilde{\varepsilon}$  and taking (2.11) into account, we find  $u = (\hat{x}_1, \hat{x}_2) \in X^2$  and  $v = (\tilde{x}_1, \tilde{x}_2) \in X^2$  satisfying (2.6) so that

$$\begin{aligned} 0 & \in \hat{\partial}g(\hat{x}_1, \hat{x}_2) + \hat{\partial}(L \operatorname{dist}(\tilde{x}_1; \Omega_1 \cap U_1) + L \operatorname{dist}(\tilde{x}_2; \Omega_2 \cap U_2)) \\ & + \tilde{\varepsilon}(B^* \times B^*). \end{aligned} \quad (2.12)$$

It follows from the constructions above that

$$\operatorname{dist}(x; \Omega_i \cap U_i) = \operatorname{dist}(x; \Omega_i), \quad i = 1, 2,$$

for all  $x$  in some neighborhoods of  $\tilde{x}_1$  and  $\tilde{x}_2$ , respectively, if  $\varepsilon$  is sufficiently small. Using properties (S4), (S3'), and the definition of  $\widehat{N}$  in (2.9), we get from (2.11) that

$$0 \in \hat{\partial}g(\hat{x}_1, \hat{x}_2) + \widehat{N}(\tilde{x}_1; \Omega_1) \times \widehat{N}(\tilde{x}_2; \Omega_2) + (\gamma - \varepsilon)(B^* \times B^*). \quad (2.13)$$

Finally invoking property (S2) and computing  $\hat{g}(\hat{x}_1, \hat{x}_2)$  by the subdifferential theory of convex analysis, we arrive at (2.2) similarly to the proof of Proposition 2.2. This justifies the prenormal structure of  $\widehat{N}$  defined in (2.9). ■

In the next section we show that *any* prenormal structure defined above allows us to obtain a general extremal principle in a fuzzy/approximate form that is the main vehicle for our applications to welfare economics in Section 5. It turns out that property (H1) alone is sufficient to derive “fuzzy” necessary conditions for optimal solutions to various optimization and related problems using an arbitrary prenormal structure  $\widehat{N}$  at points close to the reference solution. However, to get “exact” results in this direction formulated at given optimal solutions, we need more *robust* normal constructions. The latter can be obtained by using *limiting procedures* based on prenormals. We consider two kinds of such limiting procedures involving the *sequential* and *topological* Painlevé–Kuratowski upper limits defined in Section 1.

**DEFINITION 2.4.** Let  $X$  be a Banach space and let  $\widehat{N}$  be an arbitrary prenormal structure in  $X$ . We say that  $N$  defines a *sequential normal structure* in  $X$ , generated by  $\widehat{N}$ , if

$$N(\bar{x}; \Omega) := \limsup_{x \rightarrow \bar{x}} \widehat{N}(x; \Omega) \quad (2.14)$$

for every nonempty closed set  $\Omega \subset X$  and every  $\bar{x} \in X$ , where  $\limsup$  connotes the sequential Painlevé–Kuratowski upper limit. If  $\limsup$  in (2.14) is topological, then  $N$  defines the corresponding *topological normal structure* in  $X$ .

It immediately follows from (2.14) and Definition 2.1 that  $N(\bar{x}; \Omega) = \emptyset$  for  $\bar{x} \notin \Omega$  and, moreover, one may consider only  $x \in \Omega$  in (2.14). Obviously, a sequential normal structure provides generally smaller sets  $N(x; \Omega)$  than its topological counterpart. However, sequential normal structures are mostly useful in Banach spaces  $X$  whose unit dual balls  $B^* \subset X^*$  are weakly\* sequentially compact, while topological normal structures do not need such a requirement.

A remarkable example of the sequential normal structure (2.14) is generated by the prenormal cone

$$\widehat{N}(x; \Omega) := \left\{ x^* \in X^* \left| \limsup_{u \xrightarrow{\Omega} x} \frac{\langle x^*, u - x \rangle}{\|u - x\|} \leq 0 \right. \right\} \quad (2.15)$$

known also as the cone of *Fréchet* or *regular* normals to  $\Omega$  at  $x \in \Omega$ . In the finite dimensional case, the normal cone (2.14) generated by (2.15)

coincides with the one introduced by the author [25] in the form

$$N(\bar{x}; \Omega) := \operatorname{Lim\,sup}_{x \rightarrow \bar{x}} [\operatorname{cone}(x - \Pi(x; \Omega))], \quad (2.16)$$

where  $\Pi(x; \Omega)$  is the Euclidean projector of  $x$  on  $\Omega$ . This normal cone enjoys a rich calculus and numerous applications to various problems in optimization, control, stability, etc.; see the books of Mordukhovich [26] and Rockafellar and Wets [34] with many references therein and also the recent paper of Khan [21] for applications to welfare economics.

An infinite-dimensional extension of (2.16) to the case of Banach spaces first appeared in [22] in the *sequential limiting* form

$$N(\bar{x}; \Omega) := \operatorname{Lim\,sup}_{x \rightarrow \bar{x}, \varepsilon \downarrow 0} \widehat{N}_\varepsilon(x; \Omega) \quad (2.17)$$

involving the sets of  $\varepsilon$ -normals

$$\widehat{N}_\varepsilon(x; \Omega) := \left\{ x^* \in X^* \left| \limsup_{u \xrightarrow{\Omega} x} \frac{\langle x^*, u - x \rangle}{\|u - x\|} \leq \varepsilon \right. \right\}, \quad x \in \Omega, \quad (2.18)$$

with  $\widehat{N}_\varepsilon(x; \Omega) = \emptyset$  for  $x \notin \Omega$ . Note that (2.17) reduces to the normal cone (2.14) generated by (2.15) if  $X$  is an *Asplund space*, i.e., each of its separable subspaces has a separable dual. This class is sufficiently broad and convenient for applications; in particular, it contains all reflexive Banach spaces and, more generally, all spaces having a Lipschitzian Fréchet differential bump function; see, e.g., [32].

It turns out that the cone (2.15) satisfies (H1) in every Asplund space, and the corresponding normal structure (2.14) = (2.17) enjoys there full calculus and other basic properties of (2.16) at the same level of perfection as in finite dimensions; see [30] for more details and references. Moreover, the normal cone (2.17) has the following *minimality property* among any normal structures satisfying natural requirements from the viewpoint of applications to necessary optimality conditions, in particular, in the context of the generalized second welfare theorem for nonconvex economies; cf. Section 5.

**PROPOSITION 2.5.** *Let  $\Omega \subset X$  be a subset of a Banach space,  $\bar{x} \in \Omega$ , and  $\widehat{N}(\cdot; \Omega)$  satisfy the following property on  $\Omega$ :*

(M) *For any given  $x^* \in X^*$ , small  $\varepsilon > 0$ , and  $u \in \Omega \cap (\bar{x} + \varepsilon B)$  providing a local minimum to the function*

$$g(x) := \langle x^*, x - u \rangle + \varepsilon \|x - u\|, \quad x \in \Omega, \quad (2.19)$$



over  $\Omega$ , there is  $v \in \Omega \cap (\bar{x} + \varepsilon B)$  such that

$$-x^* \in \gamma B^* + \widehat{N}(v; \Omega) \quad \text{for all } \gamma > \varepsilon. \quad (2.20)$$

Then one has

$$N(\bar{x}; \Omega) \subset \limsup_{x \rightarrow \bar{x}} \widehat{N}(x; \Omega), \quad (2.21)$$

where  $N(\cdot; \Omega)$  is defined in (2.17) and  $\limsup$  signifies the sequential Painlevé–Kuratowski upper limit.

*Proof.* Taking an arbitrary  $x^* \in N(\bar{x}; \Omega)$  in (2.17), we find sequences  $\varepsilon_k \downarrow 0$ ,  $x_k \rightarrow \bar{x}$ , and  $x_k^* \xrightarrow{w^*} x^*$  with  $x_k^* \in \widehat{N}_{\varepsilon_k}(x_k; \Omega)$  for all  $k \in \mathbb{N}$ . Due to (2.18) this implies that for any  $k \in \mathbb{N}$  and any  $\eta > 0$  one has

$$\langle x_k^*, x - x_k \rangle - (\varepsilon_k + \eta) \|x - x_k\| \leq 0 \quad \text{for all } x \in \Omega \text{ near } x_k.$$

Thus,  $x_k$  gives a local minimum to a function of type (2.19) over  $\Omega$ . Employing (2.20), we get

$$x_k^* \in (2\varepsilon_k + \eta)B^* + \widehat{N}(v_k; \Omega) \quad \text{with some } v_k \in \Omega \text{ near } x_k.$$

Since  $\eta > 0$  was chosen arbitrary, the latter ensures (2.21) by passing to the limit as  $k \rightarrow \infty$ . ■

Similarly to the proof of Proposition 2.2 we can show that condition (M) holds for any  $\widehat{N}(\cdot; \Omega) = \hat{\partial}\delta(\cdot; \Omega)$  generated by a presubdifferential  $\hat{\partial}$  satisfying properties (S1) and (S2) with convex continuous functions  $g$  of type (2.19). This holds, in particular, for the prenormal cone (2.15) in Asplund spaces.

If  $X$  is not Asplund but it admits a Lipschitzian bump function differentiable with respect to some given bornology  $\beta$  (Gâteaux, Hadamard, etc.), then a sequential normal structure in  $X$  is generated via (2.14) by the so-called *viscosity  $\beta$ -normal cone* [8] that satisfies (H1) in such a  $\beta$ -smooth space. Moreover, instead of the whole set of  $\beta$ -normals, one can consider only  $\beta$ -normals of *controlled rank*; see [6].

A remarkable example of the topological normal structure in an arbitrary Banach space  $X$  is provided by the “approximate”  $G$ -normal cone of Ioffe [15] that satisfies (H1) and has the upper semicontinuity property with respect to the norm  $\times$  weak\* topology of  $X \times X^*$ . The  $G$ -normal cone reduces to (2.16) when  $X = \mathbb{R}^n$ . It possesses an exact calculus under general assumptions being smaller than Clarke’s normal cone [9] in any Banach space. On the other hand, the  $G$ -normal cone and its “nucleus” (called the “approximate normal cone” in [17]) are always bigger than the

sequential construction (2.17) and may be strictly bigger than (2.17) even in Asplund spaces admitting Fréchet smooth renorms; see more discussions in [30, Sect. 9].

The sequential and topological normal structures discussed in the above examples are generated by the corresponding subdifferentials of the indicator function. Other examples of normal structures (not cones) can be produced in the scheme (2.9) by using subdifferentials of the distance function. In what follows we are going to show that *any* normal structure from Definition 2.4 provides, under some natural assumptions, adequate “exact” versions of the extremal principle and the generalized second welfare theorem in appropriate Banach spaces.

### 3. EXTREMAL PRINCIPLE

This section is devoted to necessary optimality conditions for locally extremal points of systems of closed sets in Banach spaces obtained in fuzzy/approximate and exact/limiting forms of the extremal principle. The main goal is to derive a general “abstract” version of the fuzzy extremal principle in terms of an arbitrary prenormal structure described in Section 2 and the corresponding exact versions of this principle in terms of either sequential or topological normal structures in appropriate Banach spaces.

Let us start with the definition of a locally extremal point for a system of two sets; cf. [22, 26].

**DEFINITION 3.1.** Let  $\Omega_1$  and  $\Omega_2$  be nonempty subsets of a Banach space  $X$ . We say that  $\bar{x} \in \Omega_1 \cap \Omega_2$  is a *locally extremal point* of the set system  $\{\Omega_1, \Omega_2\}$  if for any  $\varepsilon > 0$  there are a neighborhood  $U$  of  $\bar{x}$  and a vector  $a \in X$  such that  $\|a\| \leq \varepsilon$  and

$$(\Omega_1 + a) \cap \Omega_2 \cap U = \emptyset. \quad (3.1)$$

We say that  $\{\Omega_1, \Omega_2\}$  is an *extremal system* in  $X$  if these sets have at least one locally extremal point.

An obvious example of the extremal system is provided by the pair  $\{\bar{x}, \Omega\}$ , where  $\bar{x}$  is a *boundary point* of the closed set  $\Omega \subset X$ . In general, this geometric concept of set extremality covers conventional notions of optimal solutions to various problems of scalar and vector optimization. In particular, let  $\bar{x}$  be a local solution to a standard *constrained optimization* problem

$$\text{minimize } f(x) \text{ subject to } x \in \Omega \subset X. \quad (3.2)$$

Then one can easily check that  $(\bar{x}, f(\bar{x}))$  is a locally extremal point of the set system  $\{\Omega_1, \Omega_2\}$  in  $X \times \mathbb{R}$  with  $\Omega_1 := \text{epi} f$  and  $\Omega_2 := \Omega \times \{f(\bar{x})\}$ . Indeed, for any  $\varepsilon > 0$  we satisfy the requirements of Definition 3.1 with  $a = (0, \nu)$  and  $U = O \times \mathbb{R}$ , where  $0 < \nu \leq \varepsilon$  and  $O$  is a neighborhood of the local minimizer  $\bar{x}$  in (3.2).

The following result is an abstract version of the fuzzy extremal principle in terms of an arbitrary prenormal structure  $\widehat{N}$  from Definition 2.1.

**THEOREM 3.2.** *Let  $\bar{x}$  be a locally extremal point for the system  $\{\Omega_1, \Omega_2\}$  of closed sets in a Banach space  $X$ , and let  $\widehat{N}$  be a prenormal structure in  $X$ . Then for every  $\varepsilon > 0$  there are  $x_i \in \Omega_i \cap (\bar{x} + \varepsilon B)$ ,  $i = 1, 2$ , and  $x^* \in X^*$  with  $\|x^*\| = 1$  such that*

$$x^* \in (\widehat{N}(x_1; \Omega_1) + \varepsilon B^*) \cap (-\widehat{N}(x_2; \Omega_2) + \varepsilon B^*). \quad (3.3)$$

*Proof.* We'll basically follow the proof of [29, Lemma 4.1], where the fuzzy extremal principle is derived from the fuzzy sum rule for Fréchet subdifferentials. Let us show that property (H1) ensures this result for an arbitrary prenormal structure.

Given a locally extremal point  $\bar{x}$  of the set system  $\{\Omega_1, \Omega_2\}$  and a number  $\varepsilon > 0$ , we take  $\tilde{\varepsilon} := \varepsilon/2$  and find  $a \in X$  with  $\|a\| \leq \tilde{\varepsilon}$  such that (3.1) holds for some neighborhood  $U$  of  $\bar{x}$ . We can always assume that  $\bar{x} + \tilde{\varepsilon}B \subset U$ . Let us form the function

$$f(x_1, x_2) := \|x_1 - x_2 + a\| \quad \text{for } (x_1, x_2) \in X^2. \quad (3.4)$$

It follows from (3.1) and (3.4) that  $f(\bar{x}, \bar{x}) = \|a\| \leq \tilde{\varepsilon}$  and

$$f(x_1, x_2) > 0$$

$$\text{for all } (x_1, x_2) \in Y := [\Omega_1 \cap (\bar{x} + \tilde{\varepsilon}B)] \times [\Omega_2 \cap (\bar{x} + \tilde{\varepsilon}B)].$$

One can see that  $Y$  is a complete metric space with the metric induced by norm (2.3) on  $X^2$ , and that the function  $f$  is obviously continuous on  $Y$ . So we can apply the Ekeland variational principle to the function  $f$  on the space  $Y$ . Using this result, we find  $(\bar{x}_1, \bar{x}_2) \in Y$  such that

$$f(\bar{x}_1, \bar{x}_2) \leq f(x_1, x_2) + \tilde{\varepsilon}(\|x_1 - \bar{x}_2\| + \|x_2 - \bar{x}_2\|) \quad \text{for all } (x_1, x_2) \in Y.$$

The latter implies that  $(\bar{x}_1, \bar{x}_2) \in \Omega_1 \times \Omega_2$  is a local minimizer of the function

$$g(x_1, x_2) := \|x_1 - x_2 + a\| + \tilde{\varepsilon}(\|x_1 - \bar{x}_1\| + \|x_2 - \bar{x}_2\|)$$

relative to the set  $\Omega_1 \times \Omega_2$  with  $\bar{x}_1 - \bar{x}_2 + a \neq 0$ . Now applying property (H1) of the normal structure  $\widehat{N}$  with  $\gamma := \varepsilon > \tilde{\varepsilon}$ , we find  $\tilde{x}_i \in \bar{x}_i + \tilde{\varepsilon}B$ ,  $i = 1, 2$ , and  $x^* \in X^*$  with  $\|x^*\| = 1$  such that

$$(-x^*, x^*) \in \widehat{N}(\tilde{x}_1; \Omega_1) \times \widehat{N}(\tilde{x}_2; \Omega_2) + \varepsilon(B^* \times B^*).$$

It follows from the constructions above that  $(\tilde{x}_1, \tilde{x}_2) \in \Omega_1 \times \Omega_2$  and  $\tilde{x}_i \in \bar{x} + \varepsilon B$ ,  $i = 1, 2$ . Thus we get all the relationships of the fuzzy extremal principle. ■

If  $\widehat{N}(\cdot; \Omega) = \hat{\partial}\delta(\cdot; \Omega)$  is a subdifferentially generated prenormal structure in  $X$ , then the extremal principle of type (3.3) is known to be equivalent to several versions of the subdifferential fuzzy sum rule and certain other basic results of variational analysis under additional requirements on  $\hat{\partial}$  in comparison with those listed in Proposition 2.2; cf. [16, 29, 35] for more details, proofs, and discussions. Theorem 3.2 together with Propositions 2.2 and 2.3 reveals minimal hypotheses sufficient for the validity of the fuzzy extremal principle.

Since conventional notions of optimal solutions to various constrained optimization problems can be reduced to locally extremal points of some systems of sets, Theorem 3.2 directly implies “fuzzy” necessary optimality conditions for such problems in terms of an arbitrary prenormal structure. The reader may consult with [26, 28] and their references for typical results in this direction mostly obtained in terms of the constructions (2.15)–(2.17). Another immediate consequence of the fuzzy extremal principle is the following density result that can be treated as an abstract nonconvex generalization of the celebrated Bishop–Phelps theorem on the density of support points for closed convex sets; cf. [29, Corollary 3.4; 32, Theorem 3.18].

**COROLLARY 3.3.** *Let  $\Omega$  be a closed subset of a Banach space  $X$ , and let  $\widehat{N}$  be a prenormal structure in  $X$ . Then for every boundary point  $\bar{x}$  of  $\Omega$  and every  $\varepsilon > 0$  there is  $x \in \Omega \cap (\bar{x} + \varepsilon B)$  such that  $\widehat{N}(x; \Omega) \neq \{0\}$ .*

*Proof.* It follows from Theorem 3.2 with  $\Omega_1 := \Omega$  and  $\Omega_2 := \{\bar{x}\}$ . ■

Next let us discuss sufficient conditions under which one can pass to the limit in (3.3) as  $\varepsilon \downarrow 0$  and derive the *exact* form of the extremal principle in terms of the (limiting) *normal structures*  $N(\cdot; \Omega)$  described in Definition 2.4. To furnish this for both sequential and topological normal structures, we need the following sequential assumption on the set  $\Omega$  in question formulated in terms of the corresponding prenormal structure  $\widehat{N}$  generating  $N(\cdot; \Omega)$  by virtue of (2.14).

DEFINITION 3.4. Let  $\widehat{N}$  be a prenormal structure in a Banach space  $X$ . We say that  $\Omega \subset X$  is *sequentially normally compact* (SNC) at  $\bar{x} \in \Omega$  with respect to  $\widehat{N}$  if for any sequence  $(x_k, x_k^*)$  satisfying

$$x_k^* \in \widehat{N}(x_k; \Omega), \quad x_k \rightarrow \bar{x}, \text{ and } x_k^* \xrightarrow{w^*} 0$$

one has  $\|x_k^*\| \rightarrow 0$  as  $k \rightarrow \infty$ .

The SNC property was formulated in [31] in terms of the prenormal cone (2.15) although it has been actually used earlier for furnishing limiting procedures. It obviously holds in finite dimensions and is always implied by its *topological* counterpart, where sequences  $(x_k, x_k^*)$  are replaced with nets  $(x_\alpha, x_\alpha^*)$  such that  $x_\alpha^*$  are uniformly (norm) bounded. Nice characterizations of the latter topological property are obtained by Ioffe [17] for a class of subdifferentially generated normal cones. It is proved in [17] that such a topological normal compactness property is equivalent to the *compactly epi-Lipschitzian* (CEL) property of Borwein and Strojwas [7] if  $\widehat{N}$  is generated either by the approximate subdifferential in arbitrary Banach spaces, or by the Fréchet subdifferential in Asplund spaces, or by viscosity  $\beta$ -subdifferentials in  $\beta$ -smooth spaces for any bornology  $\beta$ . Let us mention that the CEL property is *intrinsic*; i.e., it does not depend on prenormal structures as the ones in Definition 3.4. Explicit characterizations of the CEL property were recently obtained in [5] for closed convex sets in any normed (possibly incomplete) spaces.

Now we are ready to derive the limiting form of the extremal principle in terms of general sequential and topological normal structures in Banach spaces, where the SNC property is used in both cases.

THEOREM 3.5. Let  $\bar{x}$  be a locally extremal point for the system  $\{\Omega_1, \Omega_2\}$  of closed sets in a Banach space  $X$ , and let  $\widehat{N}$  be a prenormal structure in  $X$ . Assume that one of the sets  $\Omega_i$ ,  $i = 1, 2$ , is sequentially normally compact at  $\bar{x}$ . Then there is  $x^* \in B^* \setminus \{0\}$  such that

$$x^* \in N(\bar{x}; \Omega_1) \cap (-N(\bar{x}; \Omega_2)), \quad (3.5)$$

where  $N$  stands for the topological normal structure (2.14) generated by  $\widehat{N}$ . If in addition the dual ball  $B^* \subset X^*$  is weakly\* sequentially compact, then (3.5) holds with the sequential normal structure  $N$  generated by  $\widehat{N}$ .

*Proof.* Let us first prove the sequential version of the theorem assuming that the dual ball  $B^*$  is weakly\* sequentially compact in  $X^*$ . Take a sequence  $\varepsilon_k \downarrow 0$  and consider the corresponding sequences  $(x_{1k}, x_{2k}, x_k^*)$  satisfying conclusions of Theorem 3.2. We have  $x_{1k} \rightarrow \bar{x}$  and  $x_{2k} \rightarrow \bar{x}$  as  $k \rightarrow \infty$ . Since  $\|x_k^*\| = 1$  and  $B^*$  is weakly\* sequentially compact, we select a subsequence of  $\{x_k^*\}$  (without relabeling) that weakly\* converges to some

$x^* \in B^*$ . According to (3.3) there are  $x_{ik}^* \in \widehat{N}(x_{ik}; \Omega_i)$  and  $b_{ik}^* \in B^*$ ,  $i = 1, 2$ , such that

$$x_k^* = x_{1k}^* + \varepsilon_k b_{1k}^* \quad \text{and} \quad x_k^* = -x_{2k}^* + \varepsilon_k b_{2k}^* \quad \text{for all } k \in \mathbb{N}. \quad (3.6)$$

It follows from (3.6) that  $x_{1k}^* \xrightarrow{w^*} x^*$  and  $x_{2k}^* \xrightarrow{w^*} -x^*$  as  $k \rightarrow \infty$ . The latter implies, due to the definition of the sequential normal structure (2.14), that  $x^*$  satisfies (3.5).

To prove the theorem in the sequential case, it remains to show that  $x^* \neq 0$ . On the contrary, assume that  $x^* = 0$ . Thus  $x_{ik}^* \xrightarrow{w^*} 0$  for the sequences  $x_{ik}^* \in \widehat{N}(x_{ik}; \Omega_i)$ ,  $i = 1, 2$ . Since one of the sets  $\Omega_i$  (say  $\Omega_1$ ) is sequentially normally compact at  $\bar{x}$ , we get  $\|x_{1k}^*\| \rightarrow 0$ . Then (3.6) implies that  $\|x_k^*\| \rightarrow 0$ , which contradicts the condition  $\|x_k^*\| = 1$  for all  $k \in \mathbb{N}$  and justifies the theorem in the sequential case.

To prove the theorem in the general case, we follow the same procedure using the well-known fact that  $B^*$  is (topologically) weakly\* compact in arbitrary Banach spaces. Based on this fact, we conclude that the above sequence  $\{x_k^*\}$  has a weak\* cluster point  $x^* \in \text{cl}^*\{x_k^* | k \in \mathbb{N}\} \cap B^*$ . It follows from representation (3.6) with  $x_{ik}^* \in \widehat{N}(x_{ik}; \Omega_i)$ ,  $i = 1, 2$ , and the definition of the topological Painlevé–Kuratowski upper limit that  $x^*$  satisfies (3.5), where  $N$  is the topological normal structure (2.14) generated by  $\widehat{N}$ . This holds for any cluster point  $x^* \in \text{cl}^*\{x_k^* | k \in \mathbb{N}\}$ .

Finally let us show that the SNC property of one of the sets  $\Omega_i$  at  $\bar{x}$  implies that  $x^* \neq 0$  for some  $x^* \in \text{cl}^*\{x_k^* | k \in \mathbb{N}\}$ . Indeed, the opposite means that  $x^* = 0$  is the only weak\* cluster point of  $\{x_k^*\}$ . The latter yields that the whole sequence  $x_k^*$  weakly\* converges to zero. Then it follows from (3.6) that  $x_{ik}^* \xrightarrow{w^*} 0$ ,  $i = 1, 2$ , as  $k \rightarrow \infty$ . Hence  $\|x_{ik}^*\| \rightarrow 0$  for either  $i = 1$  or  $i = 2$ , which is impossible due to  $\|x_k^*\| = 1$  in (3.6). This completes the proof of the theorem. ■

Note that  $B^*$  is sequentially weakly\* compact in  $X^*$  if  $X$  is either an Asplund space or a  $\beta$ -smooth Banach space for any bornology  $\beta$ . Therefore, Theorem 3.5 contains the exact extremal principle in terms of the sequential normal structures generated by, respectively, the Fréchet sub-differential in Asplund spaces and the viscosity  $\beta$ -subdifferentials in  $\beta$ -smooth spaces via either indicator or distance functions. In the case of arbitrary Banach spaces, Theorem 3.5 implies, in particular, the exact extremal principle in terms of the (topological)  $G$ -normal cone under the *sequential* normal compactness condition. A similar observation that a sequential compactness property is sufficient to deal with a related limiting topological structure was made in [17] in the context of metric regularity.

According to the above discussions, the extremal principle of Theorem 3.5 allows us to derive “exact” necessary optimality conditions for various constrained optimization problems in terms of an abstract normal structure  $N$ . Note also that Theorem 3.5 implies that  $N(\bar{x}; \Omega) \neq \{0\}$  if  $\bar{x}$  is a boundary point of a closed set  $\Omega$  having the sequential normal compactness property at  $\bar{x}$ .

#### 4. MODELS OF WELFARE ECONOMICS

One of the objectives of this paper is to develop applications of the general extremal principle to the study of Pareto optimality in nonconvex models of welfare economics. This section is devoted to the description of the basic economic model under consideration and the corresponding concepts of Pareto optimal allocations we are going to study by means of the extremal principle.

Let  $E$  be an arbitrary Banach space that is a *commodity space* in the following model  $\mathcal{E}$  of welfare economics. The model involves  $n$  consumers with consumption sets  $C_i \subset E$  ( $i = 1, \dots, n$ ) and  $m$  firms, technological possibilities of which are represented by sets  $S_j \subset E$  ( $j = 1, \dots, m$ ). Each consumer has a *preference set*  $P_i(x)$  that consists of elements in  $C_i$  preferred to  $x_i$  by this consumer at the consumption plan  $x = (x_1, \dots, x_n) \in C_1 \times \dots \times C_n$ . Thus, the generalized preference relation is given by  $n$  arbitrary multifunctions  $P_i: C_1 \times \dots \times C_n \rightrightarrows C_i$  and does not use any ordering, utility functions, transitive relations, etc. We always assume that at least one consumer  $i_0 \in \{1, \dots, n\}$  is *nonsatiated*, i.e.,  $P_{i_0}(x) \neq \emptyset$ . For convenience we put  $\text{cl}P_i(x) := \{x_i\}$  if  $P_i(x) = \emptyset$  for some  $i$ .

To describe a link between consumers and producers in the economy  $\mathcal{E}$ , we use a general subset  $W \subset E$  of the commodity space called the *net demand constraint set* [28]. In classical models the set  $W$  consists of one element  $\{\omega\}$ , which is the initial *aggregate endowment* of scarce resources. The usage of the general constraint set  $W$  allows us to deal with the case of *uncertainty* in economic modeling when the initial endowment may not be known exactly due to, e.g., incomplete information.

**DEFINITION 4.1.** Let  $x = (x_i) = (x_1, \dots, x_n)$  and  $y = (y_j) = (y_1, \dots, y_m)$ . We say that  $(x, y) \in \prod_{i=1}^n C_i \times \prod_{j=1}^m S_j$  is a *feasible allocation* of  $\mathcal{E}$  if

$$\sum_{i=1}^n x_i - \sum_{j=1}^m y_j \in W. \quad (4.1)$$

When  $W = \{\omega\}$  with the given initial aggregate endowment  $w$ , condition (4.1) reduces to the classical “markets clear” condition. If  $E$  is a commodity space *ordered* by the closed positive cone  $E_+$  and  $W := \omega - E_+$ , market constraints (4.1) corresponds to the “implicit free disposal” of commodities.

The following generalized notions of Pareto optimal allocations studied in this paper reduce to conventional concepts of Pareto optimality (efficiency) for economic models with preference relations given by some preorder and/or utility functions; see, e.g., [2, 12, 14, 20, 24] and the references therein.

**DEFINITION 4.2.** Let  $(\bar{x}, \bar{y})$  be a feasible allocation of the economy  $\mathcal{E}$  with the property  $\bar{x}_i \in \text{cl} P_i(\bar{x})$  for all  $i = 1, \dots, n$ . We say that:

(i)  $(\bar{x}, \bar{y})$  is a *weak Pareto local optimal allocation* of  $\mathcal{E}$  if there is a neighborhood  $O$  of  $(\bar{x}, \bar{y})$  such that for every feasible allocation  $(x, y) \in O$  one has  $x_i \notin P_i(\bar{x})$  for some  $i \in \{1, \dots, n\}$ .

(ii)  $(\bar{x}, \bar{y})$  is a *Pareto local optimal allocation* of  $\mathcal{E}$  if there is a neighborhood  $O$  of  $(\bar{x}, \bar{y})$  such that for every  $(x, y) \in O$  either  $x_i \notin \text{cl} P_i(\bar{x})$  for some  $i \in \{1, \dots, n\}$  or  $x_i \notin P_i(\bar{x})$  for all  $i = 1, \dots, n$ .

(iii)  $(\bar{x}, \bar{y})$  is a *strong Pareto local optimal allocation* of  $\mathcal{E}$  if there is a neighborhood  $O$  of  $(\bar{x}, \bar{y})$  such that for every  $(x, y) \in O$  with  $(x, y) \neq (\bar{x}, \bar{y})$  one has  $x_i \notin \text{cl} P_i(\bar{x})$  for some  $i \in \{1, \dots, n\}$ .

In Section 5 we derive necessary conditions for all the three notions of Pareto optimal allocations from Definition 4.2 by reducing them to locally extremal points of some systems of sets. To furnish this, we need to impose additional assumptions that take into account the specific character of each of the Pareto optimality concepts under consideration. It is remarkable that *strong Pareto* optimal allocations for economies with ordered commodity spaces can be reduced to locally extremal points under some natural assumptions, which are *not* related to the classical interiority condition  $\text{int} E_+ \neq \emptyset$ ; see Section 5. The cases of *weak Pareto* and *Pareto* optimality are different; they require some *qualification conditions* that can be considered as appropriate generalizations of nonempty interiority assumptions for nonconvex economies with non-ordered and ordered infinite-dimensional commodity spaces. The following qualification conditions, imposed in [23, 28] for our economic model with general net demand constraints, are in line with the desirability direction condition of Mas-Colell [24] and “condition  $(\Delta)$ ” of Cornet [10] used also by Khan [21] under the name of Cornet’s constraint qualification.



DEFINITION 4.3. Let  $(\bar{x}, \bar{y})$  be a feasible allocation for  $\mathcal{E}$  and let

$$\bar{w} := \sum_{i=1}^n \bar{x}_i - \sum_{j=1}^m \bar{y}_j, \quad (4.2)$$

$$\begin{aligned} \Delta_\varepsilon := & \sum_{i=1}^n \text{cl}P_i(\bar{x}) \cap (\bar{x}_i + \varepsilon B) - \sum_{j=1}^m \text{cl}S_j \cap (\bar{y}_j + \varepsilon B) \\ & - \text{cl}W \cap (\bar{w} + \varepsilon B). \end{aligned}$$

We say that:

(i) The *net demand weak qualification (NDWQ) condition* holds at  $(\bar{x}, \bar{y})$  if there are  $\varepsilon > 0$  and a sequence  $\{e_k\} \subset E$  with  $e_k \rightarrow 0$  as  $k \rightarrow \infty$  such that

$$\Delta_\varepsilon + e_k \subset \sum_{i=1}^n P_i(\bar{x}) - \sum_{j=1}^m S_j - W \quad \text{for all large } k \in \mathbb{N}. \quad (4.3)$$

(ii) The *net demand qualification (NDQ) condition* holds at  $(\bar{x}, \bar{y})$  if there are  $\varepsilon > 0$ , a sequence  $\{e_k\} \subset E$  with  $e_k \rightarrow 0$  as  $k \rightarrow \infty$ , and a consumer index  $i_0 \in \{1, \dots, n\}$  such that

$$\Delta_\varepsilon + e_k \subset P_{i_0}(\bar{x}) + \sum_{i \neq i_0} \text{cl}P_i(\bar{x}) - \sum_{j=1}^m S_j - W \quad \text{for all large } k \in \mathbb{N}. \quad (4.4)$$

Obviously the NDWQ condition implies the NDQ one but not vice versa. When  $W = \{\omega\}$  (the markets clear) and all the production sets  $S_j$  are locally closed, the NDQ condition reduces to the “asymptotically included condition” of Jofré and Rivera [19], which directly implies (4.4) in the general case under consideration. So the sufficient conditions for the latter property presented in [18, 19] as well as those for Cornet’s constraint qualification presented in [10, 21], ensure the validity of the net demand qualification condition (4.4). Note that Cornet’s constraint qualification corresponds to (4.4) with no set  $W$ , where  $e_k = t_k e$  for some fixed  $e \in X$  and  $t_k \downarrow 0$ . The following sufficient conditions for the NDWQ and NDQ properties are proved in [23]. They involve the notion of *epi-Lipschitzian* sets introduced by Rockafellar [33]. Note that for convex sets this notion reduces to a nonempty interior.

PROPOSITION 4.4. *Let  $X$  be a Banach space and let  $(\bar{x}, \bar{y})$  be a feasible allocation for the economy  $\mathcal{E}$ . The following assertions hold:*

(i) *Assume that  $\bar{x}_i \in \text{cl} P_i(\bar{x})$  for all  $i = 1, \dots, n$ . Then the NDWQ condition is satisfied at  $(\bar{x}, \bar{y})$  if the set*

$$\Delta := \sum_{i=1}^n P_i(\bar{x}) - \sum_{j=1}^m S_j - W \quad (4.5)$$

*is epi-Lipschitzian at  $0 \in \text{cl} \Delta$ . It happens when either one among the sets  $P_i(\bar{x})$ ,  $S_j$ , and  $W$  or some of their partial combinations in (4.5) are epi-Lipschitzian at the corresponding point.*

(ii) *Assume that  $n > 1$ . The NDQ condition is satisfied at  $(\bar{x}, \bar{y})$  if there is a nonsatiated consumer  $i_0 \in \{1, \dots, n\}$  such that the set*

$$\Sigma := \sum_{i \neq i_0} \text{cl} P_i(\bar{x})$$

*is epi-Lipschitzian at the point  $\sum_{i \neq i_0} \bar{x}_i$ . It happens when either one among the sets  $\text{cl} P_i(\bar{x})$  for  $i \in \{1, \dots, n\} \setminus \{i_0\}$  or some of their partial sums are epi-Lipschitzian at the corresponding point.*

According to Proposition 4.4 we do not need to impose any assumption on preference and production sets for the fulfillment of both qualification conditions in Definition 4.3 if the net demand constraint set  $W$  is epi-Lipschitzian at the point  $\bar{w}$  in (4.2). This happens, in particular, when  $E$  is ordered and  $W = \omega - E_+$  with  $\text{int} E_+ \neq \emptyset$ . The latter covers the so-called “free-disposal Pareto optimum” studied by Cornet [11] in finite dimensions.

## 5. NECESSARY CONDITIONS FOR PARETO OPTIMAL ALLOCATIONS

This section is devoted to applications of the abstract extremal principle from Section 3 to necessary conditions for Pareto optimal allocations in the nonconvex economic model  $\mathcal{E}$  described above. We present here three basic results in this direction, which have an appropriate form of the *generalized second welfare theorem* ensuring the existence of a *common marginal equilibrium price* at Pareto optimal allocations. First we present an *approximate form* of the generalized second welfare theorem, where an equilibrium price system is formalized in terms of abstract prenormal structures from Section 2. The next result ensures an *exact form* of the second welfare theorem expressed in terms of general limiting normal

structures under the sequential normal compactness assumption on one the sets involved. These results are formulated in parallel for weak Pareto and Pareto local optimal allocations under the corresponding qualification conditions from Definition 4.3. They extend and unify previous versions of the second welfare theorem obtained in terms of specific normal structures under more restrictive assumptions. The last result of this section contains new versions of the generalized second welfare theorem for *strong* Pareto local optimal allocations in convex and nonconvex economies with ordered commodity spaces without the mentioned qualification conditions.

Let us start with an *approximate version* of the second welfare theorem that provides “fuzzy” necessary conditions for Pareto and weak Pareto optimal allocations in terms of general prenormal structures  $\widehat{N}$  from Definition 2.1. To obtain this result, we need to postulate, in addition to (H1), the following two natural properties of a prenormal structure  $\widehat{N}$  in a Banach space  $X$  that certainly hold for all reasonable constructions.

(H2) If  $\Omega \subset X$  is a linear subspace of  $X$  and  $\bar{x} \in \Omega$ , then  $\widehat{N}(\bar{x}; \Omega) = \Omega^\perp$  is a subspace orthogonal to  $\Omega$ .

(H3) For every closed subsets  $\Omega_1$  and  $\Omega_2$  of  $X$  with  $\Omega_1 \times \Omega_2 \subset X$  and every  $\bar{x}_i \in \Omega_i$ ,  $i = 1, 2$ , one has

$$\widehat{N}((x_1, \bar{x}_2); \Omega_1 \times \Omega_2) \subset \widehat{N}(\bar{x}_1; \Omega_1) \times \widehat{N}(\bar{x}_2; \Omega_2).$$

**THEOREM 5.1.** *Let  $E$  be a Banach space,  $\mathcal{E}$  an economy with the commodity space  $E$ ,  $X := E^{n+m+1}$ , and  $\widehat{N}$  a prenormal structure in  $X$  satisfying hypotheses (H1)–(H3). Considering a Pareto (weak Pareto) local optimal allocation  $(\bar{x}, \bar{y})$  of  $\mathcal{E}$  with  $\bar{w}$  defined in (4.2), we assume that the net demand qualification condition (resp. net demand weak qualification condition) holds at  $(\bar{x}, \bar{y})$ . Then for every  $\varepsilon > 0$  there are vectors  $(x, y, w) \in \Pi_{i=1}^n \text{cl} P_i(\bar{x}) \times \Pi_{j=1}^m \text{cl} S_j \times \text{cl} W$  and a nonzero price  $p^* \in E^*$  such that*

$$1 - (\varepsilon/2) \leq \|p^*\| \leq 1 + (\varepsilon/2), \quad (5.1)$$

$$-p^* \in \widehat{N}(x_i; \text{cl} P_i(\bar{x})) + \varepsilon B^* \\ \text{with } x_i \in \bar{x}_i + (\varepsilon/2)B \text{ for all } i = 1, \dots, n; \quad (5.2)$$

$$p^* \in \widehat{N}(y_j; \text{cl} S_j) + \varepsilon B^* \quad \text{with } y_j \in \bar{y}_j + (\varepsilon/2)B \text{ for all } j = 1, \dots, m; \quad (5.3)$$

$$p^* \in \widehat{N}(w; \text{cl} W) + \varepsilon B^* \quad \text{with } w \in \bar{w} + (\varepsilon/2)B. \quad (5.4)$$

*Proof.* We prove the theorem in a parallel way for Pareto and weak Pareto local optimal allocations  $(\bar{x}, \bar{y})$ . Define two closed sets in the

Banach space  $X = E^{n+m+1}$  as

$$\Omega_1 := \left\{ (x, y, w) \in X \left| \sum_{i=1}^n x_i - \sum_{j=1}^m y_j - w = 0 \right. \right\}, \quad (5.5)$$

$$\Omega_2 := \prod_{i=1}^n \text{cl} P_i(\bar{x}) \times \prod_{j=1}^m \text{cl} S_j \times \text{cl} W. \quad (5.6)$$

Let us show that  $(\bar{x}, \bar{y}, \bar{w})$  is a locally extremal point of the set system  $\{\Omega_1, \Omega_2\}$  provided that the NDWQ (resp. NDQ) condition holds. Select  $\varepsilon > 0$  so small that  $(\bar{x} + \varepsilon B, \bar{y} + \varepsilon B) \subset O$  for the given neighborhood  $O$  of  $(\bar{x}, \bar{y})$  in Definition 4.2 and that the corresponding qualification condition in Definition 4.3 is satisfied along a sequence  $\{e_k\} \subset E$ . It follows directly from Definitions 4.1 and 4.2 that  $(\bar{x}, \bar{y}, \bar{w}) \in \Omega_1 \cap \Omega_2$ . Denote  $U := (\bar{x} + \varepsilon B) \times (\bar{y} + \varepsilon B) \times (\bar{w} + \varepsilon B)$ . To justify the local extremality of  $(\bar{x}, \bar{y}, \bar{w})$  for  $\{\Omega_1, \Omega_2\}$ , it is sufficient to find a sequence  $\{a_k\} \subset X$  such that  $a_k \rightarrow 0$  as  $k \rightarrow \infty$  and

$$(\Omega_1 + a_k) \cap \Omega_2 \cap U = \emptyset \quad \text{for all large } k \in \mathbb{N}. \quad (5.7)$$

To proceed, we take a sequence  $\{e_k\} \subset E$  converging to zero for which either (4.3) or (4.4) is satisfied and put  $a_k := (0, \dots, 0, e_k) \in X$ . Assuming that (5.7) does not hold, we find  $z_k \in \Omega_2 \cap U$  with  $z_k - a_k \in \Omega_1$ . Due to the structure of sets (5.5) and (5.6) and the construction of  $a_k$ , this implies the existence of

$$x_{ik} \in \text{cl} P_i(\bar{x}) \cap (\bar{x}_i + \varepsilon B), \quad i = 1, \dots, n;$$

$$y_{jk} \in \text{cl} S_j \cap (\bar{y}_j + \varepsilon B), \quad j = 1, \dots, m,$$

and  $w_k \in \text{cl} W \cap (\bar{w} + \varepsilon B)$  such that

$$\sum_{i=1}^n x_{ik} - \sum_{j=1}^m y_{jk} - w_k + e_k = 0.$$

The latter means, by definition of the set  $\Delta_\varepsilon$ , that  $0 \in \Delta_\varepsilon + e_k$ . Due to (4.3) and (4.4) this contradicts the weak Pareto local optimality of  $(\bar{x}, \bar{y})$  in the first case and the Pareto local optimality of this allocation in the second case. Thus we establish the local extremality of  $(\bar{x}, \bar{y}, \bar{w})$  for the set system  $\{\Omega_1, \Omega_2\}$  defined in (5.5) and (5.6).

Now we apply to  $\{\Omega_1, \Omega_2\}$  the extremal principle established in Theorem 3.2 in terms of an arbitrary prenormal structure  $\widehat{N}$  in the Banach space  $X$ . According to this result, for every  $\varepsilon > 0$  there are  $u \in \Omega_1$ ,  $v = (x_1, \dots, x_n, y_1, \dots, y_m, w) \in \Omega_2$ ,

$$u^* \in \widehat{N}(u; \Omega_1), \quad v^* \in \widehat{N}(v; \Omega_2), \quad (5.8)$$

and  $x^* \in X^*$  such that

$$x^* - u^* \in (\varepsilon/2)B^*, \quad x^* + v^* \in (\varepsilon/2)B^*, \quad \|x^*\| = 1; \quad (5.9)$$

$$\begin{aligned} x_i \in \bar{x}_i + (\varepsilon/2)B, \quad i = 1, \dots, n; \quad y_j \in \bar{y}_j + (\varepsilon/2)B, \quad j = 1, \dots, m; \\ w \in \bar{w} + (\varepsilon/2)B. \end{aligned} \quad (5.10)$$

Observe that the set  $\Omega_1$  in (5.5) is a linear subspace separated in all the variables  $(x_i, y_j, w)$ . Involving hypothesis (H2) and computing the orthogonal subspace to  $\Omega_1$ , we conclude that  $u^* = (p^*, \dots, p^*, -p^*, \dots, -p^*)$  in (5.8) for some  $p^* \in E^*$ , where the “minus terms” start with the  $(n+1)$ st position.

It follows from the first relation in (5.9) that  $1 - (\varepsilon/2) \leq \|u^*\| \leq 1 + (\varepsilon/2)$ . Moreover, the sum norm (2.3) on the product  $X = E^{n+m+1}$  implies that  $\|u^*\| = \|p^*\|$  for the dual norms in  $X^*$  and  $E^*$ , respectively. Thus we arrive at (5.1). Then employing (5.8) and (5.9), we get

$$-u^* = (-p^*, \dots, -p^*, p^*, \dots, p^*) \in \widehat{N}(v; \Omega_2) + \varepsilon B_X^*. \quad (5.11)$$

Finally we observe that the set  $\Omega_2$  has the product structure in  $X$  and that  $B_X^* = (B_E^*)^{n+m+1}$  due to (2.3). This allows us to invoke hypothesis (H3) and to derive all the relations (5.2)–(5.4) from (5.10) and (5.11). Thus we end the proof of the theorem. ■

Theorem 5.1 extends an approximate version of the generalized second welfare theorem obtained in [23, 28] in terms of the prenormal cone (2.15) in Asplund spaces by using the corresponding version of the extremal principle. A similar result for the case of Pareto optimal allocations with the “markets clear” condition  $W = \{\omega\}$  was recently established by Jofré [18] in terms of subdifferentially generated prenormal structures of form (2.9) under some subdifferential requirements more restrictive than those we listed in Proposition 2.3. Observe that not all of the requirements in [18] (particularly the subdifferential sum rule) are satisfied for the Fréchet subdifferential in Asplund spaces. The proof in [18] is based on a subdifferential condition for boundary points of the sum of closed sets obtained by Borwein and Jofré [4] in Banach spaces. The latter result can be treated as an approximate version of the nonconvex separation property established by Jofré and Rivera [19] in finite dimensions as an extension of the unpublished result by Cornet and Rockafellar (1989).

Now, passing to the limit in the relations of Theorem 5.1, we derive an *exact form* of the generalized second welfare theorem for nonconvex economies in terms of abstract normal structures in Banach spaces.

**THEOREM 5.2.** *Let  $(\bar{x}, \bar{y})$  be a Pareto (weak Pareto) local optimal allocation of the economy  $\mathcal{E}$  satisfying the corresponding assumptions of Theorem 5.1. Taking a prenormal structure  $\widehat{N}$  in  $X$ , we assume in addition that either one of the sets  $\text{cl}P_i(\bar{x})$ ,  $i = 1, \dots, n$ , or  $\text{cl}S_j$ ,  $j = 1, \dots, m$ , or  $\text{cl}W$  is sequentially normally compact at the corresponding point with respect to  $\widehat{N}$ . Then there is a nonzero price  $p^* \in E^*$  satisfying*

$$-p^* \in N(\bar{x}_i; \text{cl}P_i(\bar{x})) \quad \text{for all } i = 1, \dots, n; \quad (5.12)$$

$$p^* \in N(\bar{y}_j; \text{cl}S_j) \quad \text{for all } j = 1, \dots, m; \quad (5.13)$$

$$p^* \in N(\bar{w}; \text{cl}W), \quad (5.14)$$

where  $N$  stands for the topological normal structure (2.14) generated by  $\widehat{N}$ . One can use the sequential normal structure in (5.12)–(5.14) if the dual ball  $B^* \subset E^*$  is weakly\* sequentially compact.

*Proof.* Taking a sequence  $\varepsilon_k \downarrow 0$  in the relations of Theorem 5.1, we get  $(x_k, y_k, w_k) \in \prod_{i=1}^n [\text{cl}P_i(\bar{x})] \times \prod_{j=1}^m \text{cl}S_j \times \text{cl}W$  and  $p_k^* \in E^*$  such that  $(x_k, y_k, w_k) \rightarrow (\bar{x}, \bar{y}, \bar{w})$  as  $k \rightarrow \infty$ ,  $p_k^*$  are uniformly bounded in the norm of  $E^*$ , and (5.2)–(5.4) hold. Due to the well-known results of functional analysis, there is a weak\* cluster point  $p^* \in \text{cl}^*\{p_k^* | k \in \mathbb{N}\}$  of this sequence in the case of an arbitrary Banach space  $E$ . If the unit ball  $B^*$  of  $E^*$  is weakly\* sequentially compact (as for either Asplund or  $\beta$ -smooth Banach spaces  $E$ ), then  $\{p_k^*\}$  contains a subsequence that weakly\* converges to some  $p^* \in E^*$ . Passing to the limit in (5.2)–(5.4) in both the topological and sequential cases above, we conclude that a limit point  $p^*$  in both cases satisfies relations (5.12)–(5.14) in terms of, respectively, topological and sequential structure (2.14) generated by  $\widehat{N}$ .

It remains to prove that we can choose  $p^* \neq 0$  if one of the sets  $\text{cl}P_i(\bar{x})$ ,  $\text{cl}S_j$ , and  $\text{cl}W$  is sequentially normally compact at the corresponding point. Assume for definiteness that the set  $\text{cl}W$  is sequentially normally compact at  $\bar{w}$  and that  $p^* = 0$  is the only weak\* cluster point of  $\{p_k^*\}$ . The latter yields that  $p_k^* \xrightarrow{w^*} 0$  as  $k \rightarrow \infty$ . Due to (5.4) we have

$$p_k^* + \varepsilon_k b_k^* \in \widehat{N}(w_k; \text{cl}W) \quad \text{with some } b_k^* \in B^* \text{ for all } k \in \mathbb{N},$$

and hence  $p_k^* + \varepsilon_k b_k^* \xrightarrow{w^*} 0$  as  $k \rightarrow \infty$ . This implies, by Definition 3.4, that  $\|p_k^* + \varepsilon_k b_k^*\| \rightarrow 0$  and thus  $\|p_k^*\| \rightarrow 0$  as  $k \rightarrow \infty$ . The latter clearly contradicts (5.1) and completes the proof of the theorem. ■

If  $\mathcal{E}$  is an economy with convex preference and production sets and  $N$  agrees with the normal cone of convex analysis for convex sets, then relations (5.12) and (5.13) give that  $\bar{x}_i$  minimizes the consumer's expenditure  $\langle p^*, x_i \rangle$  over  $x_i \in \text{cl}P_i(\bar{x})$  for each  $i = 1, \dots$  and that  $\bar{y}_j$  maximizes

the firm's profit  $\langle p^*, y_j \rangle$  over  $y_j \in S_j$  for each  $j = 1, \dots, m$ . This goes back to the conclusion of the classical second welfare theorem that ensures the existence of such an equilibrium price  $p^*$  under more restrictive assumptions; see, e.g., Arrow and Hahn [2] and Debreu [12]. For *nonconvex* economies  $\mathcal{E}$ , Theorem 5.2 extends and unifies various results on the generalized second welfare theorem obtained in terms of specific normal structures. In particular, it covers the results of Khan [21] (first presented in his preprint of 1987) and Cornet [11] for finite-dimensional commodity spaces when the normal cone (2.16) is used with  $W = \{\omega\}$  and  $W = \omega - \mathbb{R}_+^n$ , respectively. For the case of Asplund commodity spaces, Theorem 5.2 reduces to the corresponding versions presented in [23, 28] in terms of the normal cone (2.17). In general Banach commodity spaces, it extends Jofré's result [18] with  $W = \{\omega\}$  involving a normal structure generated by the approximate subdifferential of the distance function, where the compactly epi-Lipschitzian property is used instead of the sequential normal compactness. Hence, Theorem 5.2 also extends the result of Bonnisseau and Cornet [3] that employs a bigger Clarke's normal cone in Banach spaces using the more restrictive epi-Lipschitzian property. Recent extensions of the "exact" second welfare theorem for a general economic model with private and public goods are obtained by Flåm and Jourani [13] using a version of the extremal principle and employing an abstract notion of the subdifferential with some calculus and compactness requirements close to [18].

Note that the main difference between convex and nonconvex economies is that relations (5.12) and (5.13) in the nonconvex case provide only *first-order necessary conditions* for the consumer's expenditure minimization and the firm's profit maximization formalized in terms of the general normal structure  $N$ . Following the terminology in [11], we speak about *marginal pricing quasi-equilibrium* with respect to  $N$ . In general, when the net demand constraint set  $W$  reflects some uncertainty in the economy, the additional condition (5.14) describes restrictions on the marginal price system in order to sustain such an *uncertain* quasi-equilibrium.

Next we consider a special case of the economic model  $\mathcal{E}$  when the commodity space  $E$  is an *ordered Banach space* with the closed *positive cone*  $E_+ := \{e \in E | e \geq 0\}$ . The associated closed positive cone  $E_+^*$  of the dual space  $E^*$  has the representation

$$E_+^* := \{e^* \in E^* | e^* \geq 0\} = \{e^* \in E^* | \langle e^*, e \rangle \geq 0 \text{ for all } e \in E_+\}, \quad (5.15)$$

where the order on  $E^*$  is induced by the given one  $\geq$  on  $E$ . Let us present a consequence of Theorem 5.2 for the case of ordered commodity

spaces with the so-called *implicit free disposal of commodities*

$$W := \omega - E_+ \quad \text{for some } \omega \in W. \quad (5.16)$$

The following result provides additional information about the marginal price system  $p^*$  satisfying the conclusions (5.12)–(5.14) of the generalized second welfare theorem in the case of (5.16).

**COROLLARY 5.3.** *Let  $(\bar{x}, \bar{y})$  be a Pareto (weak Pareto) local optimal allocation of the economy  $\mathcal{E}$ . In addition to the corresponding assumptions of Theorem 5.2 we suppose that  $E$  is an ordered Banach space, that the net demand constraint set  $W$  is given by (5.16), and that the normal structure  $N$  agrees with the normal cone of convex analysis for closed convex subsets of  $E$ . Then there is a positive price system  $p^* \in E_+^* \setminus \{0\}$  satisfying the relations (5.12), (5.13), and*

$$\left\langle p^*, \sum_{i=1}^n \bar{x}_i - \sum_{j=1}^m \bar{y}_j - \omega \right\rangle = 0. \quad (5.17)$$

*Proof.* We need to show that, under the additional assumptions made, relation (5.14) implies the marginal price *positivity*  $p^* \geq 0$  and the complementary slackness condition (5.17), which economically means that the *value of excess demand* is zero at the marginal price. First let us justify (5.17). Since the set  $W$  in (5.16) is closed and convex, we get from (5.14) and the form of the normal cone of convex analysis that

$$\langle p^*, \bar{w} - \omega \rangle \geq \langle p^*, w - \omega \rangle \quad \text{for all } w \in W, \quad (5.18)$$

which implies  $\langle p^*, \bar{w} - \omega \rangle \geq 0$ . Since  $-E_+$  is a cone, we have from (5.16) that  $2(\bar{w} - \omega) \in -E_+ = W - \omega$ ; hence  $w := \omega + 2(\bar{w} - \omega) \in W$ . Substituting this  $w$  into (5.18), we arrive at the opposite inequality  $\langle p^*, \bar{w} - \omega \rangle \leq 0$  and justify (5.17) due to (4.3). To establish the price positivity, we observe that

$$\begin{aligned} N(\bar{w}; W) &= N(\bar{w} - \omega; -E_+) \\ &= \{e^* \in E^* \mid \langle e^*, e \rangle \geq \langle e^*, \omega - \bar{w} \rangle \text{ for any } e \in E_+\}, \end{aligned}$$

which ensures that  $p^* \geq 0$  for every  $p^* \in N(\bar{w}; W)$  due to (5.17) and (5.15). ■

The next corollary of Theorem 5.2 contains necessary conditions (in terms of an abstract normal structure) for Pareto and weak Pareto optimal allocations in nonconvex economies with ordered commodity spaces whose positive cone has a *nonempty interior*. The latter assumption ensures the



fulfillment of both the qualification conditions and the sequential normal compactness condition imposed in Theorem 5.2.

**COROLLARY 5.4.** *Let  $N$  be an abstract normal structure in  $X = E^{n+m+1}$  satisfying the requirements of Theorem 5.2 and such that  $N(\cdot; \Omega)$  is not bigger than the Clarke normal cone for closed subsets of  $E$ . Suppose in addition that  $E$  is an ordered Banach space with  $\text{int } E_+ \neq \emptyset$ . Then the following assertions hold:*

(i) *Given a weak Pareto local optimal allocation  $(\bar{x}, \bar{y})$  of  $\mathcal{E}$ , we assume that either the net demand constraint set  $W$  is closed near  $\bar{w}$  and*

$$W - E_+ \subset W, \quad (5.19)$$

*or one of the production sets  $S_j$  is closed near  $\bar{y}_j$  and satisfies the free-disposal condition*

$$S_j - E_+ \subset S_j. \quad (5.20)$$

*Then there is a nonzero marginal price  $p^* \in E^*$  satisfying relations (5.12)–(5.14).*

(ii) *Given a Pareto local optimal allocation  $(\bar{x}, \bar{y})$  of  $\mathcal{E}$  for  $n > 1$ , we assume that there is  $i \in \{1, \dots, n\}$  such that the  $i$ th consumer satisfies the desirability condition at  $\bar{x}$ :*

$$\text{cl } P_i(\bar{x}) + E_+ \subset \text{cl } P_i(\bar{x}). \quad (5.21)$$

*Then there is a nonzero marginal price  $p^* \in E^*$  satisfying relations (5.12)–(5.14).*

*Proof.* It is easy to observe that, for any subset  $\Omega$  of a Banach space, the inclusion  $\Omega + K \subset \Omega$  with some nonempty open cone  $K$  implies the epi-Lipschitzian property of  $\Omega$  at every  $\bar{x} \in \text{cl } \Omega$ . Therefore, each of the conditions (5.19)–(5.21) with  $\text{int } E_+ \neq \emptyset$  ensures the epi-Lipschitzian property of the corresponding set and thus, by Proposition 4.4, the fulfillment of the net demand (resp. weak) qualification condition imposed in Theorem 5.2. It is well known that the Clarke normal cone is weakly\* locally compact for epi-Lipschitzian sets; hence such sets have the sequential normal compactness property with respect to this cone. This implies the latter property for the corresponding sets in (5.19)–(5.21) with respect to any prenormal structure that is not bigger than the Clarke normal cone. So all the assumptions of Theorem 5.2 hold, and we get relations (5.12)–(5.14). ■

It turns out that for *strong Pareto* local optimal allocations in economies with ordered commodity spaces, we can establish both approximate and exact versions of the generalized second welfare theorem in terms of

abstract prenormal and normal structures with *no qualification conditions* imposed in Theorems 5.1 and 5.2. In particular, we can get an analog of Corollary 5.4 with  $\text{int } E_+ = \emptyset$ .

A part of the following theorem is obtained for Banach spaces ordered by their *generating* closed positive cones:  $E = E_+ - E_+$ . Note that this class is sufficiently large including, in particular, all *Banach lattices* (or normed complete *Riesz spaces*) whose generating positive cones typically have empty interiors.

**THEOREM 5.5.** *Let  $(\bar{x}, \bar{y})$  be a strong Pareto local optimal allocation of the economy  $\mathcal{E}$  with an ordered Banach commodity space  $E$ , and let the sets  $S_j$  and  $W$  be locally closed near  $\bar{y}_j$  and  $\bar{w}$ , respectively. Then one has the following assertions:*

(i) *Assume that  $E_+$  is generating and that either (5.19) holds, or (5.20) holds for some  $j \in \{1, \dots, m\}$ , or  $n > 1$  and there is a nonsatiated consumer  $i_0 \in \{1, \dots, n\}$  such that (5.21) holds for some  $i \in \{1, \dots, n\} \setminus \{i_0\}$ . Then for every  $\varepsilon > 0$  and every prenormal structure  $\widehat{N}$  satisfying hypotheses (H1)–(H3) in  $X = E^{n+m+1}$  there are  $(x, y, w) \in \prod_{i=1}^n \text{cl } P_i(\bar{x}) \times \prod_{j=1}^m S_j \times W$  and  $p^* \in X^*$  satisfying relations (5.1)–(5.4) in terms of  $\widehat{N}$ .*

(ii) *If in addition to (i) one of the sets  $\text{cl } P_i(\bar{x})$ ,  $i = 1, \dots, n$ , or  $S_j$ ,  $j = 1, \dots, m$ , or  $W$  is sequentially normally compact at the corresponding points with respect to  $\widehat{N}$ , then there is a marginal price  $p^* \in E^* \setminus \{0\}$  satisfying relations (5.12)–(5.14), where  $N$  stands for the topological normal structure (2.14) generated by  $\widehat{N}$ . One can use the sequential normal structure in (5.12)–(5.14) if the dual ball  $B^* \subset E^*$  is weakly\* sequentially compact.*

(iii) *All the conclusions in (i) and (ii) hold if, instead of the assumption that  $E_+$  is a generating cone, we assume that  $E_+ \neq \{0\}$  and at least two among the sets  $W$ ,  $S_j$ ,  $j = 1, \dots, m$ , and  $P_i(\bar{x})$ ,  $i = 1, \dots, n$ , satisfy the corresponding conditions in (5.19)–(5.21).*

*Proof.* Let us consider the system of two sets  $\{\Omega_1, \Omega_2\}$  defined in (5.5) and (5.6), where the closure operation for  $S_j$  and  $W$  in (5.6) can be omitted since these sets are locally closed around the points of interest; i.e., they can be assumed to be totally closed. Taking a *strong* Pareto local optimum  $(\bar{x}, \bar{y})$  of  $\mathcal{E}$ , we are going to show that  $(\bar{x}, \bar{y}, \bar{w}) \in \Omega_1 \cap \Omega_2$  is a *locally extremal point* of  $\{\Omega_1, \Omega_2\}$  if either the assumptions in (i) or those in (iii) hold. Thus, these assumptions replace the corresponding net demand qualification conditions in the proof of Theorem 5.1 for Pareto and weak Pareto optimal allocations.

First let us consider case (i) when the positive cone  $E_+$  is generating and either one of the sets  $W$ ,  $S_j$ , and  $P_i(\bar{x})$  satisfies the corresponding condition in (5.19)–(5.21). For definiteness we assume that (5.19) holds; the other two cases can be treated similarly.

It is easy to observe that  $\bar{w}$  is a boundary point of  $W$ ; otherwise we get a contradiction with Pareto optimality of  $(\bar{x}, \bar{y})$  under our underlying assumption that at least one of the consumers is nonsatiated. So we find a sequence  $e_k \rightarrow 0$  in  $E$  satisfying  $\bar{w} + e_k \notin W$  for all  $k \in \mathbb{N}$ . Due to the classical Krein-Šmulian theorem (see the survey of Abramovich *et al.* [1] for the proof and references), in any Banach space  $E$  ordered by a closed generating cone there exists a constant  $M > 0$  such that for each  $e \in E$  there are positive vectors  $u, v \in E_+$  with  $e = u - v$  and  $\max\{\|u\|, \|v\|\} \leq M\|e\|$ . This allows us to find sequences  $u_k \xrightarrow{E_+} 0$  and  $v_k \xrightarrow{E_+} 0$  giving  $e_k = u_k - v_k$ . Since  $v_k \in E_+$  and  $W - E_+ \subset W$ , we easily get

$$\bar{w} + u_k \notin W \quad \text{with } u_k \xrightarrow{E_+} 0 \text{ as } k \rightarrow \infty. \quad (5.22)$$

Now take a neighborhood  $O \subset E^{n+m}$  from the definition of the strong Pareto local optimal allocation  $(\bar{x}, \bar{y})$  and show that condition (5.7) is satisfied with the sequence of  $a_k := (0, \dots, 0, u_k) \in E^{n+m+1}$  and the neighborhood  $U := O \times E$ . This will justify the local extremality of  $(\bar{x}, \bar{y}, \bar{w})$  for  $\{\Omega_1, \Omega_2\}$ .

Assuming that (5.7) does not hold for some  $k \in \mathbb{N}$ , we find  $(x_k, y_k, w_k) \in \Omega_2$  such that  $(x_k, y_k) \in O$  and  $(x_k, y_k, w_k - u_k) \in \Omega_1$ . Due to  $u_k \in E_+$  and condition (5.19), the latter implies that

$$\sum_{i=1}^n x_{ik} - \sum_{j=1}^m y_{kj} = w_k - u_k \in W - E_+ \subset W \quad (5.23)$$

for the components of  $(x_k, y_k)$ . This means that  $(x_k, y_k)$  is a *feasible allocation* of  $\mathcal{E}$  belonging to the prescribed neighborhood of  $(\bar{x}, \bar{y})$ . Since  $(\bar{x}, \bar{y})$  is a strong Pareto optimum of  $\mathcal{E}$ , we get  $(x_k, y_k) = (\bar{x}, \bar{y})$  for all large  $k \in \mathbb{N}$ . So

$$\begin{aligned} \bar{w} + u_k &= \sum_{i=1}^n \bar{x}_i - \sum_{j=1}^m \bar{y}_j + u_k = \sum_{i=1}^n x_{ik} - \sum_{j=1}^m y_{jk} + u_k \\ &= (w_k - u_k) + u_k = w_k \in W, \end{aligned}$$

which contradicts (5.22) and proves the local extremality of  $(\bar{x}, \bar{y}, \bar{w})$  for  $\{\Omega_1, \Omega_2\}$  in case (i).

Let us prove this in case (iii) assuming for definiteness that (5.19) holds and one of the production sets (say  $S_1$ ) satisfies the free-disposal condition (5.20). Choose a sequence  $u_k \xrightarrow{E_+} 0$  with  $u_k \neq 0$  for all  $k \in \mathbb{N}$ , which is always possible due to  $E_+ \neq \{0\}$ . Now we form the sequence  $a_k := (0, \dots, 0, u_k) \in X$  and show that the extremality condition (5.7) holds along

this sequence with  $U := O \times E$ . Assuming the contrary and repeating the arguments above, we find  $(x_k, y_k, w_k) \in \Omega_2 \cap U$  satisfying (5.23). The latter implies that  $(x_k, y_k) = (\bar{x}, \bar{y})$  for all large  $k \in \mathbb{N}$  since  $(\bar{x}, \bar{y})$  is a strong Pareto local optimum. It follows from (5.23) in this case that

$$\sum_{i=1}^n x_{ik} - (y_{1k} - u_k) - \sum_{j=2}^m y_{kj} = w_k \in W \quad (5.24)$$

for all  $k \in \mathbb{N}$  sufficiently large. Due to (5.20) for  $j = 1$  we have  $y_{1k} - u_k \in S_1$ , and (5.24) ensures that  $(x_k, y_k - (u_k, 0, \dots, 0))$  is a feasible allocation of  $\mathcal{E}$  belonging to the prescribed neighborhood of the strong Pareto local optimum  $(\bar{x}, \bar{y})$ . This implies that  $y_{1k} - u_k = \bar{y}_1 - u_k = \bar{y}_1$ , i.e.,  $u_k = 0$  for all large  $k \in \mathbb{N}$ , a contradiction. Thus we have justified the local extremality of  $(\bar{x}, \bar{y}, \bar{w})$  for  $\{\Omega_1, \Omega_2\}$  in case (iii). Now we can apply the extremal principle of Theorem 3.2 to this system of sets and get all the conclusions of the theorem in the same way as proving Theorems 5.1 and 5.2. ■

As a corollary of Theorem 5.5 (and actually of Theorem 5.2 for every *weak* Pareto optimum) we get Khan's version of the generalized second welfare theorem obtained in [20] in terms of Ioffe's approximate normal cone in (5.12) and (5.13) under substantially more restrictive assumptions for an economy with a reflexive preference relation and a *lattice* structure of the commodity space. Namely, it is assumed in [20] that  $W = \omega - E_+$ , that both conditions (5.20) and (5.21) hold for *all*  $j = 1, \dots, m$  and  $i = 1, \dots, n$ , and that *every* preference and production set is epi-Lipschitzian at the corresponding point. Note that the extremal principle allow us to obtain results parallel to those in this paper for nonconvex economies with *public goods* that are also considered by Khan [20, 21].

Finally let us note that if *either one* of the conditions (5.19)–(5.21) holds in ordered Banach spaces, then the corresponding relation of (5.12)–(5.14) in terms of the sequential limiting normal cone (2.17) automatically implies the marginal price *positivity*  $p^* \in E_+^*$ ; see [23] for the proof and more discussions.

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